Dual transformations in one-dimensional classical and quantum mechanics

1998 J. Phys. A: Math. Gen. 31 L381

(http://iopscience.iop.org/0305-4470/31/20/001)

J. Phys. A: Math. Gen. **31** (1998) L381–L389. Printed in the UK PII: S0305-4470(98)91099-X

LETTER TO THE EDITOR

Dual transformations in one-dimensional classical and quantum mechanics

Laurence Mittag† and Michael J Stephen‡

† 84 Prescott Street, Cambridge, MA 02138, USA
‡ Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, USA

Received 27 January 1998

Abstract. Dual transformations in two-dimensional classical and quantum mechanical systems have been widely studied using conformal mapping techniques but one-dimensional systems have been largely ignored. In this paper we study dual transformations in one-dimensional mechanical systems, both classical and quantum mechanical, using some previously developed methods. A number of examples, mostly involving periodic motion or bound states, are presented. Dual transformations provide interesting connections between hitherto unconnected problems.

1. Introduction

Recently there has been some interest in dual transformations in one-particle classical and quantum mechanical systems [1–9]. Most investigations have been based on conformal mapping techniques useful in two-dimensional systems. One-dimensional systems have been largely ignored, except for certain radial equations occurring in higher-dimensional central force problems. In this paper we show that dual transformations can be applied in a wide variety of one-dimensional classical and quantum mechanical, one-particle systems and present a number of examples. Dual transformations provide some interesting connections between previously unconnected problems.

In the classical case we consider a particle of unit mass described by the space and time variables (x, t) and a dual particle of unit mass described by variables (y, T). These dynamical systems are connected by space and time transformations previously developed [1]

$$(x, t) \to (y, T) \qquad y = f(x) \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2. \tag{1}$$

The method is applicable when the potential energy of the first particle can be written as

$$V(x) = \lambda \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \nu \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \tag{2}$$

The dual potential is (see later)

$$U(y) = -\mu \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + \nu \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right). \tag{3}$$

In these equations λ, ν and μ are constants. In the potentials we have included terms linear in the derivatives. Such terms have not been previously considered†. The energies of the particles are μ and $-\lambda$, respectively, so that the energy equations can be written as

$$\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \lambda\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \nu\frac{\mathrm{d}y}{\mathrm{d}x} = \mu \tag{4}$$

$$\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}T}\right)^2 - \mu\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + \nu\frac{\mathrm{d}x}{\mathrm{d}y} = -\lambda. \tag{5}$$

For such dual systems the energy and the coupling constant μ and λ exchange roles and signs. To show the duality of these systems we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}T}\frac{\mathrm{d}y}{\mathrm{d}x} \tag{6}$$

and when this is substituted in (4) we obtain (5). The relation between the times t and T for the two particles can be determined from the last equation in (1), once the motion of one of the particles is known. It is interesting to note that time proceeds differently for the dual particles.

Alternatively [1] we can consider the action integral $S = \int L\,\mathrm{d}t$ with a Langrangian with a constant term added:

$$L = \frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \lambda\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \nu\frac{\mathrm{d}y}{\mathrm{d}x} + \mu. \tag{7}$$

The above transformations preserve the action

$$S = \int L\,\mathrm{d}t = \int \hat{L}\,\mathrm{d}T \qquad \hat{L}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = L. \tag{8}$$

In section 2 we study some examples defined by their transformation $y = f(x)$‡. Once this transformation is given the form of the potentials (2) and (3) is determined. Alternatively, given the potential $V(x)$, the transformation can be found by integrating (2). Some previous results have been included, namely radial equations derived from higher-dimensional central force problems.

In the quantum mechanical case we consider a particle and its dual described by the time-independent Schrödinger equation. The potentials for the two particles are of the form (2) and (3), respectively. Owing to the transformation properties of the one-dimensional Laplacian it is necessary to include a term in the Schrödinger equation proportional to $\{y, x\}$, the Schwarzian derivative of y with respect to x

$$\{y, x\} = \frac{\mathrm{d}}{\mathrm{d}x}(y''/y') - \frac{1}{2}(y''/y')^2 \tag{9}$$

where a prime indicates differentiation with respect to x. Such expressions occur in the conformal mapping of circular triangles [10], the theory of second-order linear differential equations [11] and higher WKB approximations. Then setting $\hbar = m = 1$ the dual quantum systems are

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - a\{y, x\} + \lambda\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \nu\frac{\mathrm{d}y}{\mathrm{d}x}\right]\psi = \mu\psi \tag{10}$$

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \left(\frac{1}{4} - a\right)\{x, y\} - \mu\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + \nu\frac{\mathrm{d}x}{\mathrm{d}y}\right]\varphi = -\lambda\varphi. \tag{11}$$

† It is possible to include linear derivatives $\alpha w'(z) + \alpha^* w'(z)^* \rightarrow \alpha z'(w)^* + \alpha^* z'(w)$ in the conformal two-dimensional treatment [1].

‡ A third proof can be based on the Hamilton–Jacobi equation.

μ and $-\lambda$ are again the energies of the two particles and ν and a are constants. $\{x, y\}$ is the Schwarzian derivative of x with respect to y and it is useful to note that

$$\{x, y\} = -\{y, x\}/y'^2. \tag{12}$$

The wavefunctions ψ and φ are related by

$$\psi = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{1/2} \varphi. \tag{13}$$

It is, of course, necessary to include boundary conditions in the solutions of (10) and (11) and to examine how these boundary conditions transform under the transformation $y = f(x)$. In section 3 we consider some examples and again it is useful to denote them by the transformation $y = f(x)$. The examples are mostly concerned with bound states.

2. Dual one-dimensional dynamical systems

In this section we consider some examples of dual one-dimensional classical dynamical systems where the energy of the systems are given by (3) and (4). They are conveniently denoted by their transformation $y = f(x)$†. They have been chosen to have periodic solutions if possible and in such cases $f(x)$ is given by an odd function. For most examples $\nu = 0$. Phases are chosen for convenience.

2.1. $y = \sinh x, -\infty < x, y < \infty$

A particle of unit mass moves in a particular $V(x) = \lambda \cosh^2 x$. The dual potential is $U(y) = -\mu/(1 + y^2)$, the negative of a 'witch' representing a well where a particle can be trapped if its energy $-\lambda$ is negative.

The solution for the x particle is given in terms of Jacobian elliptic functions. Let $2\lambda = \omega^2$, $2\mu = \omega^2 \cosh^2 a$, $k = \tanh a$, where ω is the small-amplitude oscillation frequency and a is the amplitude. Then

$$k' \sinh x(t) = k \,\mathrm{cn}\left(\frac{\omega t}{k'}, k\right) \tag{14}$$

where $k'^2 = 1 - k^2$. The solution of the dual system in given parametrically by

$$k' y(T) = k \cos\phi \qquad k'\omega T = E(\phi, k) \tag{15}$$

or

$$k' y(T) = k \,\mathrm{cn}\, u \qquad k'\omega T = E(u) = \int_o^u \mathrm{d}n^2\, u \,\mathrm{d}u \tag{16}$$

where $E(\phi, k)$ is an elliptic integral of the second kind and $E(u)$ is the Jacobi extension.

The relation between the time t and T in the two problems is given by

$$k'\omega T = E\left(\frac{\omega t}{k'}\right). \tag{17}$$

A similar energy equation occurs in the following problem: a unit mass moves on frictionless wire shaped like a parabola $z = \frac{1}{2}y^2$ suspended in a gravitational field.

† To avoid equations like $y = af(x/b)$, when a and b have the dimensions of length, all equations are in dimensionless form except for the time. Since the differential coefficient $\mathrm{d}T/\mathrm{d}t$ appears it is unnecessary for the time to be dimensionless.

2.2. $y = g\,\mathrm{d}x$ *or* $\sin y = \tanh x$, $-\infty < x < \infty$, $-\pi/2 < y < \pi/2$

The Gudermannian angle $g\,\mathrm{d}x$ is associated with the right-angled triangle with legs $(1, \sinh x)$ and hypotenuse $(\cosh x)$. The potential $V(x) = \lambda\,\mathrm{sech}^2\,x$. We choose λ and μ negative, $2\lambda = -\omega^2$, $2\mu = -\omega^2\,\mathrm{sech}^2\,a$, where again ω is the small oscillation frequency and a the amplitude. The solution is given in terms of elementary functions.

$$\sinh x(t) = \sinh a \sin(\omega t\,\mathrm{sech}\,a). \tag{18}$$

The frequency decreases with increasing amplitude.

The potential of the dual system is

$$-\mu\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 = -\mu\sec^2 y$$

and the particle is confined in an infinite potential well. Let $\omega = p\cosh a$ and $\alpha = g\,\mathrm{d}a$. The solution is straightforward with frequency increasing with increasing amplitude.

$$\sin y(T) = \sin\alpha\sin(pT\sec\alpha). \tag{19}$$

The time equation is found by eliminating $x(t)$ and $y(T)$ from $\sinh x = \tan y$

$$\sin pt = \frac{k'\sin\omega T}{\Delta(\omega T)} \qquad \Delta(\omega T) = \sqrt{l - k^2\sin^2\omega T} \tag{20}$$

where $k = \sin\alpha$, $k'^2 = 1 - k^2$.

2.3. $y = \sin(x/2)$, $-\pi \leqslant x \leqslant \pi$, $-1 \leqslant y \leqslant 1$

This yields an unexpected result. The potential $V(x) = (\lambda/4)\cos^2(x/2)$ differs by a constant from that of a pendulum of unit length and mass. The constant λ is chosen to be negative $\lambda = -8\omega^2$, the variable x is the angle in radians of the swing measured counterclockwise from its lowest point and the potential energy is measured from its highest point. If the maximum angle reached by the particle is $x = \alpha$ with $|\alpha| < \pi$ then only finite oscillations, not librations, occur and $\mu = -2\omega^2\cos^2(\alpha/2)$.

The solution is again given in terms of elliptic functions with modulus $k = \sin(\alpha/2)$:

$$\sin\tfrac{1}{2}x(t) = k\,\mathrm{sn}(\omega t, \alpha). \tag{21}$$

The potential of the dual system is

$$-\mu\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 = -4\mu/(1-y^2) = -2\mu((1+y)^{-1} + (1-y)^{-1}).$$

A particle is trapped between two repulsive Newtonian forces and oscillates between them. The solution is given parametrically by $(2\mu = -p^2/4)$

$$\begin{cases} y(T) = k\sin\phi \\ pT = k'E(\phi, k) \end{cases} \quad \text{or} \quad \begin{cases} y(T) = k\,\mathrm{sn}\,u \\ pT = k'E(u). \end{cases} \tag{22}$$

The time equation is $pT = k'E(\omega t)$ with $k'^2 = 1 - k^2$, $p = 8\omega k'$.

2.4. $y = \frac{1}{2}x|x|$, $-\infty < x$, $y < \infty$

This unusual transformation is both continuous and has a continuous derivative $\mathrm{d}y/\mathrm{d}x = |x|$. The potential is that of a simple harmonic oscillator and we choose $2\lambda = \omega^2$ and $2\mu = (\omega a)^2$ so that ω is the oscillation frequency and a the amplitude. The solution is $x(t) = a\sin\omega t$. The dual potential can be determined from $|x|^2 = 2|y|$. With λ and μ positive for the simple harmonic oscillator, the potential and total energy are negative in the dual case. The dual potential in $-\mu/2|y|$ which is Newtonian with an attractive inverse square form directed towards the origin. The oscillating particle passes through the singularity at the origin twice each period. The solution and time equation are

$$2y(T) = a^2|\sin\phi|\sin\phi \qquad 4\omega T = a^2(2\phi - \sin 2\phi) = a^2(2\omega t - \sin 2\omega t). \tag{23}$$

The $y(T)$ curve in each half cycle is a cycloid.

2.5. $R = r^\alpha$, $2\alpha = n + 3$, $0 \leqslant R$, $r < \infty$

This case includes all the two-dimensional central force problems, where the force is a power of the radius. If h replaces $r^2\dot{\theta}$ in the energy equation we obtain

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{h^2}{2r^2} + \lambda\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^2 = \mu. \tag{24}$$

The transformation $R = r^\alpha$, $\mathrm{d}T/\mathrm{d}t = (\mathrm{d}R/\mathrm{d}r)^2$ results in

$$\frac{1}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}T}\right)^2 + \frac{H^2}{2R^2} - \mu\left(\frac{\mathrm{d}r}{\mathrm{d}R}\right)^2 = -\lambda \tag{25}$$

with $r = R^\beta$, $2\beta = N + 3$, $H^2 = h^2/\alpha^2$ and the relation $(n+3)(N+3) = 4$. This situation has been studied extensively in the literature [1, 2] and we will not discuss it further except for two cases.

2.6. The radial energy equation for the Newtonian potential $-K/r$ is

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{1}{2}\frac{h^2}{r^2} - \frac{K}{r} = \mu. \tag{26}$$

The presence of the two terms proportional to r^{-2} and r^{-1} suggests treating this as in equation (4) by letting $\mathrm{d}y/\mathrm{d}r = r^{-1}$, $(\mathrm{d}y/\mathrm{d}r)^2 = r^{-2}$. Thus the Kepler problem has two duals determined by $R = r^{1/2}$ as in equation (24) and $y = \log r (0 < r < \infty, -\infty < y < \infty)$. We consider the second case. Reducing (26) to dimensionless form by letting $r = br'$, where b is the semi-minor axis of the elliptic orbit, results in (we omit the prime)

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{h^2}{b^6}\left(\frac{b}{r} - a\right)^2 = \frac{h^2}{b^6}f^2 \tag{27}$$

where a is the semi-major axis and $f^2 = a^2 - b^2$ is the focus length. The transformations $y = \log r$, $\mathrm{d}T/\mathrm{d}t = (\mathrm{d}y/\mathrm{d}r)^2$ changes this into

$$\left(\frac{\mathrm{d}y}{\mathrm{d}T}\right)^2 + \frac{h^2}{b^6}(b\,\mathrm{e}^y - a)^2 = \frac{h^2}{b^6}f^2. \tag{28}$$

The potential in this case consists of two exponentials and has the form of the Morse potential [13]. The integration can be performed by using a trick ascribed to Abel. Since equation (28) is a sum of squares we let

$$\frac{\mathrm{d}y}{\mathrm{d}T} = \frac{hf}{b^3}\sin u \qquad b\,\mathrm{e}^y - a = -f\cos u \tag{29}$$

and using $\mathrm{d}y = f\sin u/(a - f\cos u)\,\mathrm{d}u$ we have

$$\mathrm{d}T = \mathrm{d}y \bigg/ \left(\frac{\mathrm{d}y}{\mathrm{d}T}\right) = \frac{b^3}{h}\frac{\mathrm{d}u}{(a - f\cos u)}. \tag{30}$$

This can be integrated directly giving

$$b\,\mathrm{e}^{-y} = a + f\cos(hT/b^2). \tag{31}$$

3. Quantum mechanics

In this section we consider some dual one-particle quantum mechanical systems with the same forms (2) and (3) for the potential as in the classical system. Then the dual Schrödinger equations are given by (10) and (11). It is again useful to denote these examples by the transformation $y = f(x)$ and these examples are mostly concerned with bound states.

3.1. Harmonic oscillator $y = \frac{1}{2}x|x|$, $-\infty < x,\ y < \infty$

We choose $a = \nu = 0$, $2\lambda = \omega^2$ and (10) becomes the harmonic oscillator with eigenvalues $\mu_n = (n + \frac{1}{2})\omega$, $n = 0, 1, 2, \ldots$. The dual problem (11) has $\{x, y\} = 3/8y^2$ and is

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{3}{32y^2} - \frac{\mu}{2|y|}\right]\varphi = -\lambda\varphi \tag{32}$$

which is not the one-dimensional Coulomb problem. Setting $y = (2/\mu)u$ we get the equation

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}u^2} - \frac{3}{32u^2} - \frac{1}{|u|}\right]\varphi = -\frac{4\lambda}{\mu^2}\varphi = -\epsilon\varphi. \tag{33}$$

The bound-state eigenvalues of this problem are $\epsilon_n = 2/(n+\frac{1}{2})^2$ and the eigenfunctions follow from those of the harmonic oscillator via (13).

An alternative procedure in this case is to use the mapping $y = \frac{1}{2}x^2$, $0 < y < \infty$. The same dual equation (33) results but y can now be interpreted as a radial variable and the transformation $y = 2r/\mu$ in (32) gives

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{3}{32r^2} - \frac{1}{r}\right]\varphi = -\frac{4\lambda}{\mu^2}\varphi. \tag{34}$$

We compare this with the radial Coulomb problem in d dimensions with angular momentum ℓ

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(k-1)(k-3)}{8r^2} - \frac{1}{r}\right]R = -\epsilon R \tag{35}$$

where $k = 2\ell + d$ and $-\epsilon$ is the eigenvalue. Equations (34) and (35) match only when $k = 5/2, 3/2$ corresponding to special values of $2\ell+d$. The eigenvalues are $\epsilon_n = 2/(n+\frac{1}{2})^2$.

3.2. Coulomb problem $y = 2x/\sqrt{|x|}$; $-\infty < x, y < \infty$ *and* y *has a continuous derivative*

In (10) we choose $a = \nu = 0$, $\lambda = -1$ and it becomes the one-dimensional Coulomb problem with eigenvalues $\mu_n = -1/2(n+1)^2$, $n = 0, 1 \ldots$.

The dual problem (11) is

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{3}{8y^2} - \frac{\mu}{4}y^2\right]\varphi = \varphi \tag{36}$$

which is not the harmonic oscillator (note μ is negative).

The transformation $y = (2/|\mu|)^{1/4}u$ reduces this to

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \frac{3}{8u^2} + \frac{1}{2}u^2\right]\varphi = \left(\frac{2}{|\mu|}\right)^{1/2}\varphi = \varepsilon\varphi \tag{37}$$

which has eigenvalues $\varepsilon_n = 2(n+1)$.

An alternate procedure is to use the mapping $y = 2\sqrt{|x|}$, $0 < y < \infty$ so that y is a radial variable and putting $y = (2\omega^2/|\mu|)^{1/4}r$ equation (36) becomes

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{3}{8r^2} + \frac{1}{2}\omega^2 r^2\right]\varphi = \left(\frac{2}{|\mu|}\right)^{1/2}\omega\varphi. \tag{38}$$

This is a special case of the radial equation for the harmonic oscillator in d dimensions with angular momentum ℓ and $k = 2\ell + d = 4, 0$. This dual has been discussed by Grant and Rosner [2].

3.3. $R = r^\alpha$, $2\alpha = n + 3$

We regard x in (10) as a radial variable r and choose a central force proportional to r^n. The radial equation in d dimensions with angular momentum ℓ is

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(k-1)(k-3)}{8r^2} + \lambda r^{n+1}\right]\psi = \mu\psi \tag{39}$$

and corresponds to the mapping $R = (2/(n+3))r^{(n+3)/2}$ when we replace y by R. It should be noted that the Schwarzian derivative of any power law is proportional to r^{-2} and can be interpreted as a centrifugal term. In (10) we choose $a =^{(k-1)(k-3)}/(n^2 + 6n + 5)$ to obtain (39).

The dual problem is also a radial equation with a central force R^N with $(N+3)(n+3) = 4$. This same relation arises in two-dimensional central force problems. The dual equation is

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}R^2} + \frac{(k'-1)(k'-3)}{8R^2} - \mu\left(\frac{2R}{N+3}\right)^{N+1}\right]\varphi = -\lambda\varphi \tag{40}$$

where $k' = ((N+3)/2)(k-2) + 2$.

In the case of the Coulomb problem, as in the classical case in section 2.6, we can proceed differently. We write (39) in the form

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(k-1)(k-3)}{8r^2} - \frac{1}{r}\right]\varphi = -\epsilon\varphi \tag{41}$$

with eigenvalues $\epsilon_n = 1/(2(n + (d-3)/2)^2)$. We choose the mapping $y = \log r$, $-\infty < y < \infty$, in (10) with $a = 0$, $\lambda = \frac{1}{8}(k-1)(k-3)$, $\nu = -1$, $\mu = -\epsilon$. The Schwarzian derivative of an exponential is a constant and the dual problem to (41) is

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + (\epsilon\,\mathrm{e}^{2y} - \mathrm{e}^{y})\right]\varphi = -\frac{1}{8}(k-2)^2\varphi. \tag{42}$$

This becomes the Morse potential (10) if we set $\epsilon = \frac{1}{2}$ and regard $(k-2)^2/8$ as the eigenvalue.

3.4. $\tan y = \sinh x$, $-\infty < x < \infty$, $-\pi/2 < y < \pi/2$

We now take (10) in the form

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\lambda}{\cosh^2 x}\right]\psi = -\mu\psi \tag{43}$$

which is soluble [12] with eigenvalues $8\mu_n = [\sqrt{1+8\lambda} - (2n+1)]^2$, $n = 0, 1\ldots.$ The dual is easily obtained and is

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \left(\mu - \frac{1}{8}\right)\tan^2 y\right]\varphi = \left(\lambda - \mu - \frac{1}{4}\right)\varphi \tag{44}$$

which corresponds to a particle in an infinite potential well. For given μ the eigenvalues are $8\lambda = [\sqrt{8\mu} + (2n+1)]^2 - 1$.

3.5. WKB approximation

We now consider the semiclassical approximation to the wavefunctions ψ and φ in equations (10) and (11). In this approximation $\psi = \mathrm{e}^{\mathrm{i}S/\hbar}$ and S is expanded in powers of $\hbar$, $S = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots$.

The Schwarzian derivatives are of order $\hbar^2$ and then only contribute to S_2, so will be omitted. Then (we put $\nu = 0$)

$$\psi_{\mathrm{WKB}} = \frac{1}{R^{1/4}}\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}x\, R^{1/2}\right] \tag{45}$$

$$\varphi_{\mathrm{WKB}} = \frac{1}{Q^{1/4}}\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}y\, Q^{1/2}\right] \tag{46}$$

where

$$R = 2\left(\mu - \lambda\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right) \qquad Q = 2\left(\mu\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 - \lambda\right). \tag{47}$$

These wavefunctions are related by the transformation $y = f(x)$ and the relation (13). In the semiclassical approximation the one-dimensional harmonic oscillator and Coulomb problems are dual.

4. Conclusion

Dual transformations in mechanical systems are a rich and interesting subject in one dimension as well as two dimensions. In classical systems the form of the equations of motion are invariant under appropriate space and time transformations which leads to a duality between different problems. In the quantum mechanical case the time-independent Schrödinger equation preserves its form under appropriate space transformations and leads to a similar duality.

We are grateful to the Physics Department, Harvard University for hospitality while this work was completed.

References

[1] Mittag L and Stephen M J 1992 *Am. J. Phys.* **60** 207–11
[2] Grant A K and Rosner J L 1995 *Am. J. Phys.* **62** 310–15
[3] Bateman D S, Boyd C and Dutta-Roy B 1992 *Am. J. Phys.* **60** 833–6
[4] Wu H and Sprung D W L 1995 *Am. J. Phys.* **63** 564
[5] Pradhan P 1995 *Am. J. Phys.* **63** 664
[6] Rockmore R M 1975 *Am. J. Phys.* **43** 29–32
[7] Faure R 1953 *C. R. Acad. Sci., Paris* **237** 603–5
[8] Arnold V I 1990 *Not. Am. Math. Soc.* **37** 144
[9] Hojman S A, Chayet S, Nunez D and Roque M A 1991 *J. Math. Phys.* **32** 1491
[10] Nehari Z 1952 *Conformal Mapping* (New York: McGraw-Hill) p 199
[11] Hille E 1969 *Lectures on Ordinary Differential Equations* (Reading, MA: Addison-Wesley) p 647
[12] Landau L D and Lifshitz E M 1997 *Quantum Mechanics* 3rd edn (Oxford: Pergamon)